Reduction to limiting algebras

Theorem Let (X, Δ) be a log pair of dimension n and let $f: X \to Z$ be a projective morphism where Z is affine and normal. Let k be a positive integer such that $D = k(K_X + \Delta)$ is Cartier. Assume:

1) $K_X + \Delta$ plt
2) $S = \lfloor \Delta \rfloor$ irreducible
3) $\exists\, G \in |D|$ st $S \not\subset \operatorname{Supp}(G)$
4) $\Delta - S \sim_{\mathbb{Q}, Z} A + B$ $S \not\subset \operatorname{Supp}(B)$
5) $-(K_X + \Delta)$ is ample.
6) Real MMP holds in dim $n-1$

Then the restricted algebra R_S is f.g.

Reminder R_S is the restricted algebra, i.e. $\mathrm{Im}\,(R \to R|_S)$

Main difficulty R_S is not a geometric algebra.

(i.e. something of the form $\bigoplus_m H^0(S, mD|_S)$)

Lemma 1 (X, Δ) klt X smooth

D $\mathbb{Q}$-div. Let $f : X \to Z$

proj. morphism Z affine,

let $\sigma \in H^0(X, L)$.

Let $S = Z(\sigma)$.

Suppose $D \leq S + \Delta$.

Then $\sigma \in H^0(X, L \otimes \mathcal{J}(D))$.

Rmk $\sigma \in H^0(X, L \otimes \mathcal{O}(-S))$

$$= H^0(X, L \otimes \mathcal{J}(S))$$

<u>Pf</u> Let $g : Y \to X$ be a log resolution of $(X, D+\Delta)$

$$g^*D - g^*S \leq g^*\Delta$$

$$\lfloor g^*D \rfloor - g^*S \leq \lfloor g^*\Delta \rfloor$$

$$g^*\sigma \in H^0(Y, g^*L(-g^*S))$$

$$\subseteq H^0(Y, g^*L(-g^*S + K_{Y/X} - \lfloor g^*\Delta \rfloor))$$

$$\subseteq H^0(Y, g^*L(K_{Y/X} - \lfloor g^*D \rfloor))$$

$$\Rightarrow \sigma \in H^0(X, L \otimes \mathcal{J}(D)) . \qquad \square$$

Lemma 2 Let (Y, Γ) be a smooth log pair, $\pi : Y \longrightarrow Z$.

Let m be a positive integer st $m(K_Y + \Gamma)$ is Cartier.

$$c_1(L) = m(K_Y + \Gamma)$$

Assume:

1) (Y, Γ) plt
2) $T = \lfloor \Gamma \rfloor$ irreducible
3) $\Gamma - T \sim_{\mathbb{Q}} A + B$

Suppose there is $H \geq 0$, $\mathrm{Supp}(T) \not\subseteq H$, st

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$$\mathrm{Im}\left(H^0(Y, L^{\otimes s}(H)) \to H^0(T, L^{\otimes s}(H)|_T)\right)$$

contains

$$\mathrm{Im}\left(H^0(T, L^{\otimes s}|_T) \to H^0(T, L^{\otimes s}(H)|_T)\right)$$

Then

$$H^0(Y, L) \twoheadrightarrow H^0(T, L|_T).$$

Pf

$$K_Y + T + \underbrace{(1-\varepsilon)(\Gamma - T) + \varepsilon A + \varepsilon B}_{\geq 0}$$

εA ample

is plt for small ε.

$\Rightarrow$ We may assume

$$K_Y + \Gamma = K_Y + T + A + B.$$

Suppose $\sigma \in H^0(T, L')$. $S = Z(\sigma)$

By assumption $\exists\, G_s \sim s\, c_1(L) + H$

st $G_s' = sS + H'$

$$N \overset{\text{def}}{=} c_1(L) - K_Y - T$$

$$N \sim_{\mathbb{Q}} (m-1)(K_Y + \Gamma) + A + B$$

$$\Theta = \frac{m-1}{ms} G_s + B$$

$$N - \Theta \sim_{\mathbb{Q}} A - \frac{m-1}{ms} H$$

By Nadel:

$$H^1(Y, \mathcal{O}_Y(K_Y+N)\otimes \mathcal{J}(\Theta)) = 0.$$

Recall $\mathcal{J}(\Theta)|_T \supseteq \mathcal{J}(\Theta|_T)$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & \mathcal{J} \arrow[r] \arrow[d] & \mathcal{J}(T,\Theta') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}(Y,\Theta)(-T) \arrow[r] \arrow[ur] & \mathcal{J}(Y,\Theta) \arrow[r] & \mathcal{J}(Y,\Theta)|_T \arrow[r] & 0
\end{tikzcd}$$

$$H^0(Y, L\otimes\mathcal{J}) \twoheadrightarrow H^0(T, L'\otimes \mathcal{J}(T,\Theta'))$$

$$\Theta' - S \leq B' + \frac{m-1}{ms} H'$$

$$\Theta' \leq S + B' + \frac{m-1}{ms} H'$$

$(Y, T+A+B)$ is plt $\Rightarrow$

$(T, B' + \frac{m-1}{ms} H')$ is klt

$\Rightarrow \quad \sigma \in H^0(T, L' \otimes \mathcal{O}(\Theta'))$

<u>Lemma 3</u> Let (Y, Γ) be a smooth log pair. $Y \to Z$

$H = lA$ $\quad$ A very ample

Let $k(K_Y + \Gamma)$ be Cartier.

Choose $m > 0$.

$L = \mathcal{O}_Y(mk(K_Y+\Gamma))$

Assume that:

1) Γ contains T with coeff 1.

2) $\exists\, m_0 \ni G_0 \in |m_0 k(K_Y+\Gamma)|$

Supp(G_0) does not contain any lc center of $K_Y + \lceil \Gamma \rceil$

$\cancel{\text{Let } \Theta = (\Gamma - T)|_T}$

$\cancel{(K_Y+\Gamma)|_T = K_T + \Theta}$

Suppose H does not contain T

then

$$\mathrm{Im}\left(H^0(Y, L(H)) \to H^0(T, L(H)|_T)\right)$$

contains

$$\mathrm{Im}\left(H^0(T, L|_T) \to H^0(T, L(H)|_T)\right)$$

Def Let $\Delta = \sum a_i \Delta_i$.

$$\langle \Delta \rangle = \sum b_i \Delta_i$$

$$b_i = \begin{cases} a_i & \text{if } 0 < a_i < 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4 Let (X, Δ) log pair.

$\exists \; g : Y \longrightarrow X$ st if

we write

$$K_Y + \Gamma = g^*(K_X + \Delta) + E$$

Then no two components of $\langle \Gamma \rangle$ intersect.

<u>Def</u> (X, Δ) log pair.

$k(K_X + \Delta)$ Cartier.

$$mk(K_X + \Delta) = M_m + F_m.$$

The <u>log mobile</u> part $mk(K_X + \Delta_m)$ of $mk(K_X + \Delta)$ is defined by

$$\Delta_m = \max\left(\Delta - \frac{F_m}{mk}, 0\right)$$

<u>Remark</u> $mk(K_X + \Delta_m)$ is <u>not</u> necessarily mobile.

<u>Lemma</u> Let (X, Δ) log pair.

Then:

1) $0 \leq \Delta_m \leq \Delta$

2) M_m is the mobile part of $mk(K_X + \Delta_m)$

3) No component of Δ_m is fixed

4) $\Delta_\bullet$ convex.

Lemma 5 Let (Y, Γ) be a smooth log pair.

$\pi : Y \to Z$ Z normal affine.

$k(K_Y + \Gamma)$ is Cartier.

$m > 0$. Assume

1) (Y, Γ) plt
2) $T = \lfloor \Gamma \rfloor$ irreducible
3) $\Gamma - T \sim_{\mathbb{Q}} A + B$
4) no two components of $\langle \Gamma \rangle$ intersect
5) $\mathrm{Fix}(mk(K_Y + \Gamma)) \cup \mathrm{Supp}\, \Gamma$ is SNC.

Then there is a sequence of blow ups $h_m : Y_m \to Y$ along smooth centers, which are strata of $\mathrm{Fix}(mk(K_Y+\Gamma))$ $\cup\ \Gamma$.

Furthermore write

$$K_{Y_m} + \Gamma_m' = h_m^*(K_Y+\Gamma) + F_m$$

Set $mk(K_{Y_m}+\Gamma_m)$ to be the log mobile part of $mk(K_{Y_m}+\Gamma_m')$.

$\Theta_m = (\Gamma_m - T)|_T$. Let N_m be the mobile part of $mk(K_Y+\Gamma_m)$, M_m the mobile part of $mk(K_T+\Theta_m)$

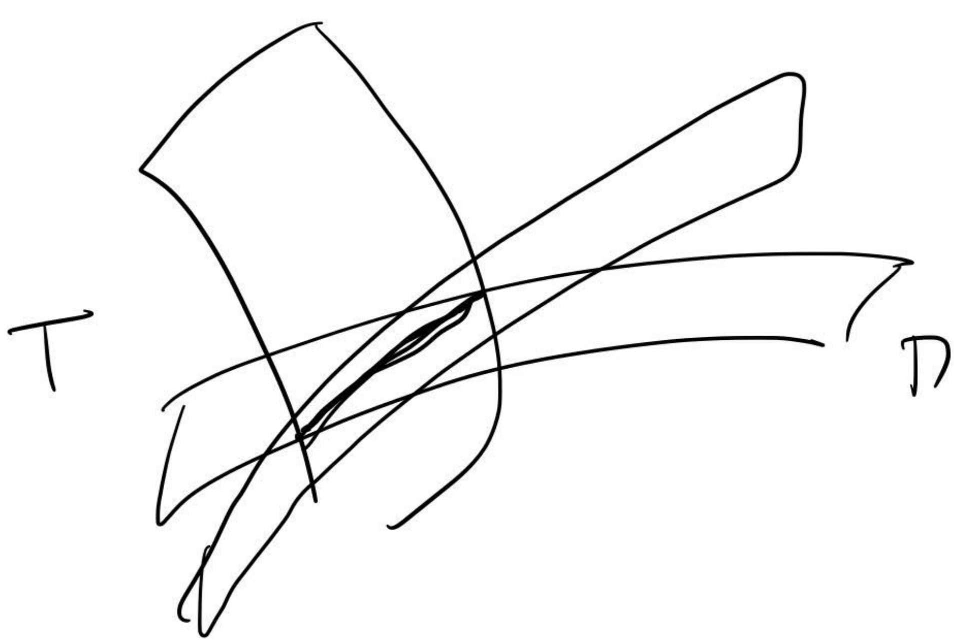
Then:

① $N_m \big|_T = M_m$

② $|N_m|_T = |M_m|$

Pf By hypothesis no two components of $\langle \Gamma \rangle$ intersect.

The lc centers of $K_Y + \lceil \Gamma \rceil$ which are contained in the base locus of $|mk(K_Y+\Gamma)|$ must be components of $T \cap \langle \Gamma \rangle$.



Blow these up.

$h_n : Y_n \to Y$ so that the base locus N_m contains no lc center of $K_{Y_n} + \lceil \Gamma_n' \rceil$.

$\Rightarrow$ contains no lc center of

$$K_{Y_n} + \lceil \Gamma_n \rceil .$$

$\exists\, F \geq 0$ exceptional st $A_n = h_n^* A - F$ is ample

$$\Gamma_n - T \sim_{\mathbb{Q}} (h_n^* A - F) + (\Gamma_n - T - h_n^* A + F)$$

$$\sim_{\mathbb{Q}} A_n + B_n$$

Thanks to Lemma 3 + lemma 2 we may lift sections:

$$H^0(Y_m, \mathcal{O}_{Y_m}(-k(K_Y + \Pi_m))) \twoheadrightarrow H^0(T, \mathcal{O}_T(-k(K_T + \underline{\underline{\Theta_m}})))$$

We may assume the same happens to their mobile parts.

Def Let R be an algebra given by B.

R is **limiting** if $\exists\ \Delta_m$ st

1) $B_m = mK(K_X + \Delta_m)$
2) $\lim \Delta_m = \Delta$ exists.
3) $K_X + \Delta$ is klt.

Theorem Let (X, Δ) log pair.

$D = k(K_X + \Delta)$ Cartier.

1) $K_X + \Delta$ qlt
2) $S = \lfloor \Delta \rfloor$ irreducible
3) $\exists\, G_0 \in |m_0 D|$

   $S \not\subseteq \operatorname{Supp}(G_0)$
4) $\Delta - S \sim_{\mathbb{Q}} A + B$

$\exists\, g_m : Y_m \to X$ st

$$K_{Y_m} + \Gamma_m' = g_m^*(K_X + \Delta) + E_m$$

then

1) the strict transform of S is independent of m.

2) $N_n|_T = M_n$

$|N_n|_T = |M_n|$

3) $\Theta_\bullet$ is convex

4) The restricted algebra R_S is fg $\Leftrightarrow$ the limiting algebra R of the $\Theta_\bullet$ is fg. $\square$

$(X, D = S + B)$ log variety.

$S = \lfloor D \rfloor \qquad B = \{D\}$

1) $K_X + D$ plt
2) $-(K_X + D)$ is nef and big $/ Z$
3) $D \in \mathcal{P}_n \qquad n \in \mathbb{N}$

Let $g : Y \longrightarrow X$ be log resolution

$$K_Y + S_Y + A = g^*(K_X + S + B)$$

$\lfloor A \rfloor \leq 0 \qquad S_Y$ proper transform of S

By inversion of adjunction S is normal

$K_S + \mathrm{Diff}_S(B)$ is plt.

$g|_S : S_Y \longrightarrow S$ birational contraction.

$$K_{S_Y} + \mathrm{Diff}_{S_Y}(A) = g_S^*\left(K_S + \mathrm{Diff}_S(B)\right)$$

<u>Note</u>: $\mathrm{Diff}_{S_Y}(A) = A|_{S_Y}$ because

Y is smooth

(This is true by requiring Y terminal only)

$\mathrm{Diff}_S(B) \in \mathcal{P}_n$

$K_S + \mathrm{Diff}_S(B)$ is n-complemented by hypothesis

$\Rightarrow$ $K_{S_Y} + \mathrm{Diff}_{S_Y}(A)$ is n-complemented

$\Rightarrow$ $\exists\, \mathrm{Diff}_{S_Y}(A)^+$

$$\exists\ \Theta \in \left| -nK_{S_Y} - \lfloor (n+1)\,\mathrm{Diff}_{S_Y}(A) \rfloor \right|$$

such that

$$n\,\mathrm{Diff}_{S_Y}(A)^+ = \lfloor (n+1)\,\mathrm{Diff}_{S_Y}(A) \rfloor + \Theta$$

By KVV

$$R^1 h_*\big(\mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor)\big)$$

$$=$$

$$R^1 h_*\big(\mathcal{O}_Y(K_Y + \lceil -(n+1)(K_Y + S_Y + A) \rceil)\big) =$$

$$=$$

$$0$$

There is a sequence

$$0 \to \mathcal{O}_Y(-nK_Y - (n+1)S_Y - \lfloor (n+1)A \rfloor)$$
$$\to \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor)$$
$$\to \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)A \rfloor|_{S_Y}) \to 0$$

$\Rightarrow$

$$H^0(Y, \mathcal{O}_Y(-nK_Y - nS_Y - \lfloor (n+1)A \rfloor))$$

$$\twoheadrightarrow H^0(S_Y, \mathcal{O}_{S_Y}(-nK_{S_Y} - \lfloor (n+1)A \rfloor|_{S_Y}))$$

He skipping:

$$0 \to \mathcal{O}_X(-nK_X - (n+1)S - \lfloor (n+1)B \rfloor))$$

$$\to \mathcal{O}_X(-nK_X - nS - \lfloor (n+1)B \rfloor))$$

$$\to \mathcal{O}_S(-nK_S - \lfloor (n+1)B \rfloor|_S) \to 0$$

$$H'(-nK_X - (n+1) - \lfloor (n+1)B \rfloor))$$

$$=$$

$$H'(K_X + \lceil -(n+1)(K_X + D) \rceil) = 0$$

$\Rightarrow$

$$\exists\ \Xi \in \left| -nK_Y - nS_Y - \lfloor (n+1)A \rfloor \right|$$

such that $\Xi|_{S_Y} = \Theta$

Set: $A^+ = \frac{1}{n}\left(\lfloor (n+1)A \rfloor + \Xi\right)$

$$n(K_Y + S_Y + A^+) \sim 0$$

$$(K_Y + S_Y + A^+)|_{S_Y} = K_{S_Y} + \mathrm{Diff}_{S_Y}(A)^+$$

A^+ could have negative coefficients. So set $B^+ = g_* A^+$

$$n(K_X + S + B^+) \sim 0$$

$$(K_X + S + B^+)|_S = K_S + \mathrm{Diff}_S(B)^+$$

Assume by contradiction

K_X+S+B^+ not lc.

$\Rightarrow$ $K_X+S+B+\alpha(B^+-B)$ not lc for some $\alpha<1$

<u>but</u> $K_X+S+B+\alpha(B^+-B)$ is nef and big over Z

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